Math 255A Lecture 24 Notes

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1 Duality and Weak Topologies

1.1 The weak topology

Definition 1.1. Let F and G be two vector spaces over $K = \mathbb{R}$ or $\mathbb{C}$, and suppose $\langle \cdot, \cdot \rangle : F \times G \to K$ sending $(x, y) \mapsto \langle x, y \rangle$ is a bilinear form. The form is said to define a **duality** between F and G if

1. If $\langle x, y \rangle = 0$ for all $y \in G$, then $x = 0$.
2. If $\langle x, y \rangle = 0$ for all $x \in F$, then $y = 0$.

Example 1.1. Let F be a Banach space B, and let $G = B^*$. Then F and G are in duality by Hahn-Banach.

Definition 1.2. The locally convex topology in F defined by the seminorms $x \mapsto |\langle x, y \rangle|$ for $y \in G$ is called the **weak topology** in F and is denoted by $\sigma(F, G)$.

We also have a weak topology $\sigma(G, F)$ in G. What are open sets in $\sigma(F, G)$? A set $O \subseteq F$ is open in $\sigma(F, G)$ iff for all $x_0 \in O$, there exists $\varepsilon > 0$ and $y_1, \dots, y_N \in G$ such that $\{x \in F : |\langle x - x_0, y_j \rangle < \varepsilon \, \forall 1 \leq j \leq N\} \subseteq O$.

$\sigma(F, G)$ and $\sigma(G, F)$ are Hausdorff topologies.

1.2 Continuity and convergence in the weak topology

Lemma 1.1. *A linear form $L : F \to K$ is continuous for $\sigma(F, G)$ if and only if there exists a unique $y \in G$ such that $L(x) = \langle x, y \rangle$ for all $x \in F$.*

Proof. ($\Longleftarrow$): This follows immediately from the definition of the topology.

($\Longrightarrow$): L is continuous for $\sigma(F, G)$ iff there exist $y_1, \dots, y_N$ and $C > 0$ such that $|L(x)| \leq C \sum_{j=1}^N |\langle x, y_j \rangle|$ for $x \in F$. We get that $\langle x, y_1 \rangle = \cdots = \langle x, y_N \rangle = 0 \implies L(x) = 0$ so that $L(x) = \sum_{j=1}^N \alpha_j \langle x, y_j \rangle = \left\langle x, \sum_{j=1}^N \alpha_j y_j \right\rangle$.[1] $\square$

[1] The proof that L is a linear combination of these forms follows from a problem on Homework 1.

Definition 1.3. Let (x_n) be a seuqence in F, and let $x \in F$. We say that $x_n \to x$ in $\sigma(F, G)$ (or **converges weakly**) if for all $y \in G$, $\langle x_n, y\rangle \to \langle x, y\rangle$.

Proposition 1.1. *Let $F = B$ be a Banach space, and let $x_n \to x$ in $\sigma(B, B^*)$. Then (x_n) is bounded: $\|x_n\| \leq C$ for $n = 1, 2, \dots$ and $\|x\| \leq \liminf_{n\to\infty} \|x_n\|$.*

Proof. For all $\xi \in B^*$, $\langle x_n, \xi\rangle$ is bounded, so by Banach Steinhaus, there exists some $C > 0$ such that $|\langle x_n, \xi\rangle| \leq C\|\xi\|$ for $n = 1, 2, \dots$. So $\|x_n\| \leq C$.

Since $|\langle x_n, \xi\rangle| \leq \|\xi\|\|x_n\|$, $|\langle x, \xi\rangle| \leq \|\xi\| \liminf_{n\to\infty} \|x_n\|$. So $\|x\| \leq \liminf_{n\to\infty} \|x_n\|$. $\square$

Example 1.2. Let $\xi_j \in \mathbb{R}^n$ be such that $|\xi_j| \to \infty$. Then, for $\varphi \in L^2$, $\varphi_n(x) := e^{ix\cdot\xi_j}\varphi(x)$ satisfies $\|\varphi_j\|_{L^2} = \|\varphi\|$, and $\varphi_j \to 0$ in $\sigma(L^2, L^2)$. Indeed, for $f \in L^2$,

$$\langle \varphi_j, f\rangle = \int e^{ix\cdot\xi_j} \underbrace{\varphi(x) f(x)}_{\in L^1} \, dx \xrightarrow{j\to\infty} 0$$

by Riemann-Lebesgue.

1.3 Closed and convex sets in the weak topology

Proposition 1.2. *Let B be a Banach space, and let $C \subseteq B$ be convex and nonempty. Then C is closed in $\sigma(B, B^*)$ if and only if C is closed in the usual (strong) sense.*

Proof. ($\implies$): If C is closed in $\sigma(B, B^*)$, then C^c is open in $\sigma(B, B^*)$, so C^c is open in the strong sense. So C is closed in the strong sense.

($\impliedby$): Let $C \subseteq B$ be convex and strongly closed. We claim that C^c is open in $\sigma(B, B^*)$. Le $x_0 \notin C$. By he geometric Hahn-Banach theorem, there exists a continuous linear form f on B such that $\inf_{x\in C}(\mathrm{Re}(f(x)) - \mathrm{Re}(f(x_0)) > 0$. Thus, there exists an $\alpha \in \mathbb{R}$ such that $\mathrm{Re}(f(x_0)) < \alpha < \mathrm{Re}(f(x))$ for $x \in X$. The set $N = f^{-1}(\{z : \mathrm{Re}(z) < a\})$ is open in $\sigma(B, B^*)$ (as f is continuous on $\sigma(B, B^*)$), $x_0 \in N$, and $N \cap C = \varnothing$. It follows that C^c is weakly open. So C is weakly closed. $\square$

Here is a fact to help with intuition for what the weak topology is like.

Proposition 1.3. *Let B be an infinite dimensional Banach space, and let $S = \{x \in B : \|x\| = 1\}$ be the unit sphere. The closure of S in $\sigma(B, B^*)$ is $\{x \in B : \|x\| \leq 1\}$.*

Proof. The closed ball $\{x : \|x\| \leq 1\}$ is convex, so it is weakly closed, and $\overline{S}^{\sigma(B,B^*)} \subseteq \{x : \|x\| \leq 1\}$. On the other hand, let $\|x_0\| < 1$. We check that any neighborhood U of x_0 in $\sigma(B, B^*)$ meets S. We can assume that $U = \{x : |\langle x - x_0, \xi_j\rangle| < \varepsilon\ \forall\}$ with $\xi_j \in B^*$. Notice that $\bigcap_{j=1}^N \ker(\xi_j) \neq \{0\}$. Let $y_0 \neq 0 \in \bigcap_{j=1}^\infty \ker(\xi_j)$. Then $x_0 + \lambda y_0 \in U$ for all λ, so the function $g(\lambda) = \|x_0 + \lambda y_0\|$ for $\lambda \geq 0$ is continuous and goes to ∞ at ∞. Since $g(0) < 1$, we get a λ such that $x_0 + \lambda y_0 \in S \cap U$. $\square$

1.4 The weak* topology

Definition 1.4. Let B be a Banach space. The weak topology $\sigma(B^*, B)$ is called the **weak* topology**.

Remark 1.1. The weak * topology on B^* is weaker than the weak topology $\sigma(B^*, B^{**})$.

Next time, we will prove the following theorem.

Theorem 1.1 (Banach-Alaoglu)**.** *The closed unit ball* $U = \{\xi \in B^* : \|\xi\|_{B^*} \leq 1\}$ *is compact in* $\sigma(B^*, B)$.